

THE M_α -DELTA INTEGRAL ON TIME SCALES

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ABSTRACT. In this paper, we define the M_α -delta integral and investigate the relation between the M_α and M_α -delta integrals on time scales.

1. Introduction and preliminaries

Throughout this paper, $I_0 = [a, b]$ is a compact interval in \mathbb{R} . Let D be a finite collection of interval-point pairs $\{(I_i, \xi_i)\}_{i=1}^n$, where $\{I_i\}_{i=1}^n$ are non-overlapping subintervals of I_0 and let δ be a positive function on I_0 . We say that $D = \{(I_i, \xi_i)\}_{i=1}^n$ is

- (1) a δ -fine McShane partition of I_0 if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in I_0$ for all $i = 1, 2, \dots, n$ and $\cup_{i=1}^n I_i = I_0$,
- (2) a δ -fine M_α -partition of I_0 for a constant $\alpha > 0$ if it is a δ -fine McShane partition of I_0 and satisfying the condition

$$\sum_{i=1}^n \text{dist}(\xi_i, I_i) < \alpha,$$

where $\text{dist}(\xi_i, I_i) = \inf\{|t - \xi_i| : t \in I_i\}$,

- (3) a δ -fine Henstock partition of I_0 if $\xi_i \in I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, 2, \dots, n$ and $\cup_{i=1}^n I_i = I_0$.

We introduce some concepts related to the notion of time scales. A time scale \mathbb{T} is any closed nonempty subset of \mathbb{R} . For each $t \in \mathbb{T}$, we define the forward jump operator $\sigma(t)$ by

$$\sigma(t) = \inf\{z \in \mathbb{T} : z > t\}$$

and the backward jump operator $\rho(t)$ by

$$\rho(t) = \sup\{z \in \mathbb{T} : z < t\}$$

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where $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say the t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ is defined by $\mu(t) = \sigma(t) - t$, and the backward graininess function $\nu(t)$ is defined by $\nu(t) = t - \rho(t)$.

For $a, b \in \mathbb{T}$, we define the time scale interval in \mathbb{T} by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

A pair $\delta = (\delta_L, \delta_R)$ of two real-valued functions on $[a, b]_{\mathbb{T}}$ is a Δ -gauge on $[a, b]_{\mathbb{T}}$ if $\delta_L(t) > 0$ on $(a, b)_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$, and $\delta_R(t) \geq \mu(t)$ for each $t \in [a, b)_{\mathbb{T}}$.

A collection $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n$ of tagged intervals is a δ -fine M_α -partition of $[a, b]_{\mathbb{T}}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_{\mathbb{T}} = [a, b]_{\mathbb{T}}$, $[t_{i-1}, t_i]_{\mathbb{T}} \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$, $\xi_i \in [a, b]_{\mathbb{T}}$ for each $i = 1, 2, \dots, n$, and $\sum_{i=1}^n \text{dist}(\xi_i, I_i) < \alpha$.

For a M_α -partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n$, we write

$$f(\mathcal{P}) = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}),$$

whenever $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$.

2. The M_α and M_α -delta integrals

DEFINITION 2.1. A function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is M_α -delta integrable (or M_α^Δ -integrable) on $[a, b]_{\mathbb{T}}$ if there exists a number A such that for each $\epsilon > 0$ there exists a Δ -gauge δ on $[a, b]_{\mathbb{T}}$ such that

$$\left| f(\mathcal{P}) - A \right| < \epsilon$$

for every δ -fine M_α -partition \mathcal{P} of $[a, b]_{\mathbb{T}}$. The number A is called the M_α^Δ -integral of f on $[a, b]_{\mathbb{T}}$, and we write $A = (M_\alpha^\Delta) \int_a^b f$.

Recall that $f : [a, b] \rightarrow \mathbb{R}$ is M_α -integrable on $[a, b]$ if there exists a number A such that for each $\epsilon > 0$ there exists a gauge $\delta : [a, b] \rightarrow \mathbb{R}^+$ on $[a, b]$ such that

$$\left| f(\mathcal{P}) - A \right| < \epsilon$$

for every δ -fine M_α -partition \mathcal{P} of $[a, b]$.

Let $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a function on $[a, b]_{\mathbb{T}}$, and let $\{(a_k, b_k)\}_{k=1}^\infty$ be the sequence of intervals contiguous to $[a, b]_{\mathbb{T}}$ in $[a, b]$.

Define a function $f^* : [a, b] \rightarrow \mathbb{R}$ on $[a, b]$ by

$$f^*(t) = \begin{cases} f(a_k) & \text{if } t \in (a_k, b_k) \text{ for some } k. \\ f(t) & \text{if } t \in [a, b]_{\mathbb{T}}. \end{cases}$$

It is well-known [9] that $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is McShane delta(or M_Δ)-integrable on $[a, b]_{\mathbb{T}}$ if and only if $f^* : [a, b] \rightarrow \mathbb{R}$ is McShane(or M)-integrable, and $(M_\Delta) \int_a^b f = (M) \int_a^b f^*$.

LEMMA 2.2. *Suppose that f is M_α^Δ -integrable on $[a, b]_{\mathbb{T}}$. Let $\epsilon > 0$ and let $\delta = (\delta_L, \delta_R)$ be a gauge on $[a, b]_{\mathbb{T}}$ such that*

$$\left| f(\mathcal{P}) - (M_\alpha^\Delta) \int_a^b f \right| < \epsilon$$

for each δ -fine M_α -partition \mathcal{P} of $[a, b]_{\mathbb{T}}$.

Assume that $\mathcal{D} = \{(\xi_i, [u_i, v_i])\}_{i=1}^n$ is a δ -fine partial M_α -partition of $[a, b]_{\mathbb{T}}$, $u_i \leq t_i \leq v_i$, $t_i \in [a, b]_{\mathbb{T}}$, $i = 1, 2, \dots, n$ and $\mathcal{D}_1 = \{(\xi_i, [u_i, t_i])\}_{i=1}^n$, $\mathcal{D}_2 = \{(\xi_i, [t_i, v_i])\}_{i=1}^n$.

Then

$$\left| f(\mathcal{D}_1) - \sum_{i=1}^n (M_\alpha^\Delta) \int_{u_i}^{t_i} f \right| \leq 3\epsilon$$

and

$$\left| f(\mathcal{D}_2) - \sum_{i=1}^n (M_\alpha^\Delta) \int_{t_i}^{v_i} f \right| \leq 3\epsilon.$$

Proof. Denote $I = \{1, 2, \dots, n\}$, $I_1 = \{i \in I \mid \xi_i \leq t_i\}$, $I_2 = \{i \in I \mid \xi_i > t_i\}$,

$$\mathcal{D}_1^1 = \{(\xi_i, [u_i, t_i])\}_{i \in I_1}, \quad \mathcal{D}_1^2 = \{(\xi_i, [u_i, t_i])\}_{i \in I_2},$$

$$\mathcal{D}_2^1 = \{(\xi_i, [t_i, v_i])\}_{i \in I_1}, \quad \mathcal{D}_2^2 = \{(\xi_i, [t_i, v_i])\}_{i \in I_2},$$

Then $\mathcal{D}_1^1, \mathcal{D}_2^2$ are δ -fine partial M_α -partition of $[a, b]_{\mathbb{T}}$. By Saks-Henstock Lemma, we have

$$\left| f(\mathcal{D}_1^1) - \sum_{i \in I_1} (M_\alpha) \int_{u_i}^{t_i} f \right| \leq \epsilon \dots \dots (1)$$

$$\left| f(\mathcal{D}_2^2) - \sum_{i \in I_2} (M_\alpha) \int_{t_i}^{v_i} f \right| \leq \epsilon \dots \dots (2)$$

Since $\{(\xi_i, [u_i, v_i])\}_{i \in I_2}$ is a δ -fine partial M_α -partition of $[a, b]_{\mathbb{T}}$, we have

$$\left| \sum_{i \in I_2} f(\xi_i)(v_i - u_i) - \sum_{i \in I_2} (M_\alpha^\Delta) \int_{u_i}^{v_i} f \right| \leq \epsilon \dots \dots (3)$$

From (2) and (3), we have

$$\begin{aligned} \left| f(\mathcal{D}_1^2) - \sum_{i \in I_2} (M_\alpha^\Delta) \int_{u_i}^{t_i} f \right| &= \left| \sum_{i \in I_2} f(\xi_i)(t_i - u_i) - \sum_{i \in I_2} (M_\alpha^\Delta) \int_{u_i}^{t_i} f \right| \\ &= \left| \left(\sum_{i \in I_2} f(\xi_i)(v_i - u_i) - \sum_{i \in I_2} f(\xi_i)(v_i - t_i) \right) \right. \\ &\quad \left. - \left(\sum_{i \in I_2} (M_\alpha^\Delta) \int_{u_i}^{v_i} f - \sum_{i \in I_2} (M_\alpha^\Delta) \int_{t_i}^{v_i} f \right) \right| \\ &\leq \left| \sum_{i \in I_2} f(\xi_i)(v_i - u_i) - \sum_{i \in I_2} (M_\alpha^\Delta) \int_{u_i}^{v_i} f \right| \\ &\quad + \left| \sum_{i \in I_2} f(\xi_i)(v_i - t_i) - \sum_{i \in I_2} (M_\alpha^\Delta) \int_{t_i}^{v_i} f \right| \leq 2\epsilon. \end{aligned}$$

Hence, we have

$$\begin{aligned} \left| f(\mathcal{D}_1) - \sum_{i=1}^n (M_\alpha^\Delta) \int_{u_i}^{t_i} f \right| &= \left| f(\mathcal{D}_1^1) + f(\mathcal{D}_1^2) - \sum_{i \in I_1} (M_\alpha^\Delta) \int_{u_i}^{t_i} f - \sum_{i \in I_2} (M_\alpha^\Delta) \int_{u_i}^{t_i} f \right| \\ &\leq \left| f(\mathcal{D}_1^1) - \sum_{i \in I_1} (M_\alpha^\Delta) \int_{u_i}^{t_i} f \right| + \left| f(\mathcal{D}_1^2) - \sum_{i \in I_2} (M_\alpha^\Delta) \int_{u_i}^{t_i} f \right| \leq 3\epsilon. \end{aligned}$$

Similarly, we have

$$\left| f(\mathcal{D}_2) - \sum_{i=1}^n (M_\alpha^\Delta) \int_{t_i}^{v_i} f \right| \leq 3\epsilon.$$

□

LEMMA 2.3. Let $f, \epsilon > 0$ and $\delta = (\delta_L, \delta_R)$ be as in the lemma 2.2.

Assume that $\mathcal{D} = \{(\xi_i, [u_i, v_i])\}_{i=1}^n$ is a δ -fine partial M_α -partition of $[a, b]_{\mathbb{T}}$, and $u_i \leq s_i < t_i \leq v_i, s_i, t_i \in [a, b]_{\mathbb{T}}, i = 1, 2, \dots, n$. Let $\mathcal{D}' = \{(\xi_i, [s_i, t_i])\}_{i=1}^n$.

Then

$$\left| f(\mathcal{D}') - (M_\alpha^\Delta) \int_{\mathcal{D}'} f \right| \leq 6\epsilon$$

where $(M_\alpha^\Delta) \int_{\mathcal{D}'} f = \sum_{i=1}^n (M_\alpha^\Delta) \int_{s_i}^{t_i} f$.

Proof. By lemma 2.2, we have

$$\left| \sum_{i=1}^n f(\xi_i)(t_i - u_i) - \sum_{i=1}^n (M_\alpha^\Delta) \int_{u_i}^{t_i} f \right| \leq 3\epsilon$$

and

$$\left| \sum_{i=1}^n f(\xi_i)(s_i - u_i) - \sum_{i=1}^n (M_\alpha^\Delta) \int_{u_i}^{s_i} f \right| \leq 3\epsilon.$$

Hence, we have

$$\begin{aligned} & \left| \sum_{i=1}^n f(\xi_i)(t_i - s_i) - \sum_{i=1}^n (M_\alpha^\Delta) \int_{s_i}^{t_i} f \right| \\ & \leq \left| \sum_{i=1}^n f(\xi_i)(t_i - u_i) - \sum_{i=1}^n (M_\alpha^\Delta) \int_{u_i}^{t_i} f \right| \\ & \quad + \left| \sum_{i=1}^n f(\xi_i)(s_i - u_i) - \sum_{i=1}^n (M_\alpha^\Delta) \int_{u_i}^{s_i} f \right| \leq 6\epsilon. \end{aligned}$$

□

THEOREM 2.4. *If $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is M_α^Δ -integrable on $[a, b]_{\mathbb{T}}$, then $f^* : [a, b] \rightarrow \mathbb{R}$ is M_α -integrable on $[a, b]$. In that case*

$$(M_\alpha) \int_a^b f^* = (M_\alpha^\Delta) \int_a^b f.$$

Proof. Let $[a, b]_{\mathbb{T}}^{rs} = \{w_k\}_{k \in \mathbb{N}}$ be the set of all right-scattered points of $[a, b]_{\mathbb{T}}$, and let $\epsilon > 0$. Then there is a Δ -gauge $\delta = (\delta_L, \delta_R)$ on $[a, b]_{\mathbb{T}}$ such that

- (1) $|f(\mathcal{P}) - (M_\alpha^\Delta) \int_a^b f| < \epsilon$ for each δ -fine M_α -partition \mathcal{P} of $[a, b]_{\mathbb{T}}$,
- (2) $w_k + \delta_R(w_k) = \sigma(w_k)$, for $k = 1, 2, 3, \dots$,
- (3) $t + \delta_R(t) \in [a, b]_{\mathbb{T}}$ for $t \in [a, b]_{\mathbb{T}}^{rd}$, where $[a, b]_{\mathbb{T}}^{rd}$ is the set of all right-dense points of $[a, b]_{\mathbb{T}}$,
- (4) $\delta_L(\sigma(w_k)) < \min \left\{ \frac{\epsilon}{2^k(|f(w_k)| + |f(\sigma(w_k))| + 1)}, \frac{\sigma(w_k) - w_k}{2} \right\}$, $k = 1, 2, \dots$,
- (5) $t - \delta_L(t) \in [a, b]_{\mathbb{T}}$ for $t \in [a, b]_{\mathbb{T}}^{ld}$, where $[a, b]_{\mathbb{T}}^{ld}$ is the set of all left-dense points of $[a, b]_{\mathbb{T}}$.

Define a gauge $\tilde{\delta} = (\tilde{\delta}_L, \tilde{\delta}_R)$ on $[a, b]$ by

- (6) $\tilde{\delta}_L(t) = \delta_L(t)$, $\tilde{\delta}_R(t) = \delta_R(t)$ if $t \in [a, b]_{\mathbb{T}}$ and $\tilde{\delta}_L(t) = \frac{t - w_k}{2}$, $\tilde{\delta}_R(t) = \sigma(t) - t$ if $t \in (w_k, \sigma(w_k))$, $k = 1, 2, \dots$.

Suppose that $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ is a $\tilde{\delta}$ -fine M_α -partition of $[a, b]$.

Put $A_1 = \{i \mid \xi_i = \sigma(w_k) \text{ and } (w_k, \sigma(w_k)) \cap [t_{i-1}, t_i] \neq \emptyset \text{ for some } k \in \mathbb{N}\}$, $A_2 = \{1, 2, \dots, n\} \setminus A_1$.

From (4), we see that for each $i \in A_1$ there is a unique $k_i \in \mathbb{N}$ such that $\xi_i = \sigma(w_{k_i})$, and $(w_{k_i}, \sigma(w_{k_i})) \cap [t_{i-1}, t_i] \neq \emptyset$.

Put $A_{11} = \{i \in A_1 \mid [t_{i-1}, t_i] \subseteq [w_{k_i}, \sigma(w_{k_i}))\}$
 $A_{12} = \{i \in A_1 \mid [t_{i-1}, t_i] \not\subseteq [w_{k_i}, \sigma(w_{k_i}))\}$.

Let

$$\mathcal{P}_1 = \{(\xi, [t_{i-1}, t_i])\}_{i \in A_2} \cup \{(t_{i-1}, [t_{i-1}, t_i])\}_{i \in A_{11}} \\ \cup \{(t_{i-1}, [t_{i-1}, \sigma(w_{k_i}))]\}_{i \in A_{12}} \cup \{(\sigma(w_{k_i}), [\sigma(w_{k_i}), t_i])\}_{i \in A_{12}}.$$

Then \mathcal{P}_1 is clearly a $\tilde{\delta}$ -fine M_α -partition of $[a, b]$ and from (4) we have

$$\begin{aligned} |f^*(\mathcal{P}_1) - f^*(\mathcal{P})| &\leq \sum_{i \in A_{11}} |f(w_{k_i}) - f(\sigma(w_{k_i}))|(t_i - t_{i-1}) \\ &\quad + \sum_{i \in A_{12}} |f(w_{k_i}) - f(\sigma(w_{k_i}))|(\sigma(w_{k_i}) - t_{i-1}) \\ &\leq \sum_{k=1}^{\infty} \left(\sum_{\substack{i \in A_{11} \\ k_i=k}} (|f(w_{k_i})| + |f(\sigma(w_{k_i}))|)(t_i - t_{i-1}) \right. \\ &\quad \left. + \sum_{\substack{i \in A_{12} \\ k_i=k}} (|f(w_{k_i})| + |f(\sigma(w_{k_i}))|)(\sigma(w_{k_i}) - t_{i-1}) \right) \\ &< \epsilon \dots \dots (7) \end{aligned}$$

For the simplicity of notation, we rewrite \mathcal{P}_1 as $\mathcal{P}_1 = \{(\eta_i, [u_i, v_i])\}_{i=1}^m$.

Then (8) If $\eta_i = \sigma(w_k)$ for some $k \in \mathbb{N}$, then $[u_i, v_i] \cap (w_k, \sigma(w_k)) = \emptyset$.

Put $I = \{1, 2, \dots, m\}$ and $W = \{w_k \mid (w_k, \sigma(w_k)) \cap [u_i, v_i] \neq \emptyset \text{ and } (w_k, \sigma(w_k)) \setminus [u_i, v_i] \neq \emptyset \text{ for some } i \in I\}$. Reorder $\{w_k\}_{k \in \mathbb{N}}$ so that $W = \{w_1, w_2, \dots, w_l\}$ and $w_1 < w_2 < \dots < w_l$. Put $J = \{1, 2, \dots, l\}$.

If $u_i \notin [a, b]_{\mathbb{T}}$, then there is a unique $p_i \in J$ such that $u_i \in (w_{p_i}, \sigma(w_{p_i}))$. If $v_i \notin [a, b]_{\mathbb{T}}$, then there is a unique $q_i \in J$ such that $v_i \in (w_{q_i}, \sigma(w_{q_i}))$. Now separate I as follow;

$$\begin{aligned}
 I_1 &= \{i \in I \mid \eta_i \in [a, b]_{\mathbb{T}}, u_i \in [a, b]_{\mathbb{T}}, v_i \in [a, b]_{\mathbb{T}}\} \\
 I_2 &= I \setminus I_1, \quad I_2^j = \{i \in I_2 \mid [u_i, v_i] \cap (w_j, \sigma(w_j)) \neq \emptyset\}, j \in J. \\
 I_{21} &= \{i \in I_2 \mid \eta_i \in [a, b]_{\mathbb{T}}, u_i \in [a, b]_{\mathbb{T}}, v_i \notin [a, b]_{\mathbb{T}}\}, I_{21}^j = I_{21} \cap I_2^j, j \in J. \\
 I_{22} &= \{i \in I_2 \mid \eta_i \in [a, b]_{\mathbb{T}}, u_i \notin [a, b]_{\mathbb{T}}, v_i \in [a, b]_{\mathbb{T}}\}, I_{22}^j = I_{22} \cap I_2^j, j \in J. \\
 I_{23} &= \{i \in I_2 \mid \eta_i \in [a, b]_{\mathbb{T}}, u_i \notin [a, b]_{\mathbb{T}}, v_i \notin [a, b]_{\mathbb{T}}, p_i = q_i\}, I_{23}^j = I_{23} \cap I_2^j, \\
 & \qquad \qquad \qquad j \in J. \\
 I_{24} &= \{i \in I_2 \mid \eta_i \in [a, b]_{\mathbb{T}}, u_i \notin [a, b]_{\mathbb{T}}, v_i \notin [a, b]_{\mathbb{T}}, p_i \neq q_i\}, I_{24}^j = I_{24} \cap I_2^j, \\
 & \qquad \qquad \qquad j \in J.
 \end{aligned}$$

From the choice of $\tilde{\delta}$, we have that if $\eta_i \notin [a, b]_{\mathbb{T}}$, then $u_i \notin [a, b]_{\mathbb{T}}$, and $\eta_i \in (w_{p_i}, \sigma(w_{p_i}))$.

Put $I_{25} = \{i \in I_2 \mid \eta_i \notin [a, b]_{\mathbb{T}}\}$, $I_{25}^j = I_{25} \cap I_2^j$, $j \in J$.

Since $\{(\eta_i, [u_i, v_i])\}_{i \in I_1}$ is a δ -fine partition M_α -partition of $[a, b]_{\mathbb{T}}$, we have by Saks-Henstock lemma,

$$\left| \sum_{i \in I_1} f^*(\eta_i)(u_i - v_i) - \sum_{i \in I_1} (M_\alpha^\Delta) \int_{u_i}^{v_i} f \right| \leq \epsilon \dots \dots (9)$$

From (2) and (3), we see that $\{(\eta_i, [u_i, \sigma(w_{q_i})])\}_{i \in B}$ is a δ -fine partial M_α -partition of $[a, b]_{\mathbb{T}}$ for each $B \subseteq I_{21}$. Hence by lemma 2.2,

$$\left| \sum_{i \in I_{21}} f^*(\eta_i)(w_{q_i} - u_i) - \sum_{i \in I_{21}} (M_\alpha^\Delta) \int_{u_i}^{w_{q_i}} f \right| \leq 3\epsilon \dots \dots (10)$$

where we use the convention $(M_\alpha^\Delta) \int_{u_i}^{w_{q_i}} f = 0$ if $w_{q_i} = u_i$.

$$\left| \sum_{i \in B} f^*(\eta_i)(\sigma(w_{q_i}) - w_{q_i}) - \sum_{i \in B} (M_\alpha^\Delta) \int_{w_{q_i}}^{\sigma(w_{q_i})} f \right| \leq 3\epsilon \dots \dots (11)$$

for each $B \subseteq I_{21}$.

Similarly, we have

$$\left| \sum_{i \in I_{22}} f^*(\eta_i)(v_i - \sigma(w_{p_i})) - \sum_{i \in I_{22}} (M_\alpha^\Delta) \int_{\sigma(w_{p_i})}^{v_i} f \right| \leq 3\epsilon \dots \dots (12)$$

$$\left| \sum_{i \in B} f^*(\eta_i)(\sigma(w_{p_i}) - w_{p_i}) - \sum_{i \in B} (M_\alpha^\Delta) \int_{w_{p_i}}^{\sigma(w_{p_i})} f \right| \leq 3\epsilon \dots \dots (13)$$

for each $B \subseteq I_{22}$.

If $i \in I_{23}$, then $[u_i, v_i] \subseteq [w_{p_i}, \sigma(w_{p_i})]$.

If $i \in I_{24}$, then $p_i + 1 = q_i$ and $w_{p_i} < u_i < \sigma(w_{p_i}) \leq w_{q_i} < v_i < \sigma(w_{q_i})$.

For $B \subseteq I_{24}$, put $B_0 = \{i \in B \mid p_i \text{ is odd}\}$, $B_e = \{i \in B \mid p_i \text{ is even}\}$. From the choice of δ , $\{(\eta_i, [w_{p_i}, \sigma(w_{p_i})])\}_{i \in B_0}$ is a δ -fine partial M_α -partition of $[a, b]_{\mathbb{T}}$. Hence by lemma 2.2 and lemma 2.3,

$$\left| \sum_{i \in B_0} f^*(\eta_i)(w_{q_i} - \sigma(w_{p_i})) - \sum_{i \in B_0} (M_\alpha^\Delta) \int_{\sigma(w_{p_i})}^{w_{q_i}} f \right| \leq 6\epsilon \dots \dots (14)$$

$$\left| \sum_{i \in B_0} f^*(\eta_i)(\sigma(w_{p_i}) - w_{p_i}) - \sum_{i \in B_0} (M_\alpha^\Delta) \int_{w_{p_i}}^{\sigma(w_{p_i})} f \right| \leq 3\epsilon \dots \dots (15)$$

$$\left| \sum_{i \in B_0} f^*(\eta_i)(\sigma(w_{q_i}) - w_{q_i}) - \sum_{i \in B_0} (M_\alpha^\Delta) \int_{w_{q_i}}^{\sigma(w_{q_i})} f \right| \leq 3\epsilon \dots \dots (16)$$

Similarly, we have

$$\left| \sum_{i \in B_e} f^*(\eta_i)(w_{q_i} - \sigma(w_{p_i})) - \sum_{i \in B_e} (M_\alpha^\Delta) \int_{\sigma(w_{p_i})}^{w_{q_i}} f \right| \leq 6\epsilon \dots \dots (17)$$

$$\left| \sum_{i \in B_e} f^*(\eta_i)(\sigma(w_{p_i}) - w_{p_i}) - \sum_{i \in B_e} (M_\alpha^\Delta) \int_{w_{p_i}}^{\sigma(w_{p_i})} f \right| \leq 3\epsilon \dots \dots (18)$$

$$\left| \sum_{i \in B_e} f^*(\eta_i)(\sigma(w_{q_i}) - w_{q_i}) - \sum_{i \in B_e} (M_\alpha^\Delta) \int_{w_{q_i}}^{\sigma(w_{q_i})} f \right| \leq 3\epsilon \dots \dots (19)$$

From (14) to (19), we have

$$\left| \sum_{i \in I_{24}} f^*(\eta_i)(w_{q_i} - \sigma(w_{p_i})) - \sum_{i \in I_{24}} (M_\alpha^\Delta) \int_{\sigma(w_{p_i})}^{w_{q_i}} f \right| \leq 12\epsilon \dots \dots (20)$$

$$\left| \sum_{i \in B} f^*(\eta_i)(\sigma(w_{p_i}) - w_{p_i}) - \sum_{i \in B} (M_\alpha^\Delta) \int_{w_{p_i}}^{\sigma(w_{p_i})} f \right| \leq 6\epsilon$$

for $B \subseteq I_{24} \dots (21)$

$$\left| \sum_{i \in B} f^*(\eta_i)(\sigma(w_{q_i}) - w_{q_i}) - \sum_{i \in B} (M_\alpha^\Delta) \int_{w_{q_i}}^{\sigma(w_{q_i})} f \right| \leq 6\epsilon$$

for $B \subseteq I_{24} \dots (22)$

If $i \in I_{25}$, then $u_i \notin [a, b]_{\mathbb{T}}$, $\eta_i \in (w_{p_i}, \sigma(w_{p_i}))$ and $[u_i, v_i] \subseteq [w_{p_i}, \sigma(w_{p_i})]$. Put $I_{241}^j = \{i \in I_{24}^j \mid p_i = j\}$, $I_{242}^j = \{i \in I_{24}^j \mid q_i = j\}$, $j \in J$, and put $I_{241} = \cup_{j \in J} I_{241}^j$, $I_{242} = \cup_{j \in J} I_{242}^j$.

From (9), (10), (12) and (20), we have

$$\begin{aligned}
 f^*(\mathcal{P}_1) &= \sum_{i \in I} f^*(\eta_i)(v_i - u_i) \\
 &\leq \sum_{i \in I_1} (M_\alpha^\Delta) \int_{u_i}^{v_i} f + \epsilon + \sum_{i \in I_{21}} (M_\alpha^\Delta) \int_{u_i}^{w_{q_i}} f + 3\epsilon \\
 &\quad + \sum_{i \in I_{22}} (M_\alpha^\Delta) \int_{\sigma(w_{p_i})}^{v_i} f + 3\epsilon + \sum_{i \in I_{24}} (M_\alpha^\Delta) \int_{\sigma(w_{p_i})}^{w_{q_i}} f \\
 &\quad + 12\epsilon + \sum_{i \in I_{21}} f^*(\eta_i)(v_i - w_{q_i}) + \sum_{i \in I_{22}} f^*(\eta_i)(\sigma(w_{p_i}) - u_i) \\
 &\quad + \sum_{i \in I_{23}} f^*(\eta_i)(v_i - u_i) + \sum_{i \in I_{24}} f^*(\eta_i)(\sigma(w_{p_i}) - u_i) \\
 &\quad + \sum_{i \in I_{24}} f^*(\eta_i)(v_i - w_{q_i}) + \sum_{i \in I_{25}} f^*(\eta_i)(v_i - u_i) \\
 &= \left[\sum_{i \in I_1} (M_\alpha^\Delta) \int_{u_i}^{v_i} f + \sum_{i \in I_{21}} (M_\alpha^\Delta) \int_{u_i}^{w_{q_i}} f + \sum_{i \in I_{22}} (M_\alpha^\Delta) \int_{\sigma(w_{p_i})}^{v_i} f \right. \\
 &\quad \left. + \sum_{i \in I_{24}} (M_\alpha^\Delta) \int_{\sigma(w_{p_i})}^{w_{q_i}} f + 19\epsilon \right] + \left[\sum_{j \in J} \left(\sum_{i \in I_2^j} f^*(\eta_i)(v_i - w_{q_i}) \right. \right. \\
 &\quad \left. \left. + \sum_{i \in I_{22}^j} f^*(\eta_i)(\sigma(w_{p_i}) - u_i) + \sum_{i \in I_{23}^j} f^*(\eta_i)(v_i - u_i) \right. \right. \\
 &\quad \left. \left. + \sum_{i \in I_{241}^j} f^*(\eta_i)(\sigma(w_j) - u_i) + \sum_{i \in I_{242}^j} f^*(\eta_i)(v_i - w_j) \right. \right. \\
 &\quad \left. \left. + \sum_{i \in I_{25}^j} f^*(\eta_i)(v_i - u_i) \right] = (I) + (II) \dots \dots (23)
 \end{aligned}$$

For each $j \in J$, choose $r_j \in I_2^j$ such that $f^*(\eta_{r_j}) = \max\{f^*(\eta_i) | i \in I_2^j\}$. Then $r_j \in I_2$ for each $j \in J$.

If $r_j \in I_{23}$, then $[u_{r_j}, v_{r_j}] \subseteq [w_{p_{r_j}}, \sigma(w_{p_{r_j}})]$ and $[u_{r_j}, v_{r_j}] \cap [w_j, \sigma(w_j)] \neq \emptyset$. Hence, $p_{r_j} = j$ and $\{(\eta_{r_j}, [w_j, \sigma(w_j)])\}_{r_j \in I_{23}}$ is a δ -fine partial M_α -partition of $[a, b]_{\mathbb{T}}$. Hence, we have

$$\left| \sum_{r_j \in I_{23}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) - \sum_{r_j \in I_{23}} (M_\alpha^\Delta) \int_{w_j}^{\sigma(w_j)} f \right| \leq \epsilon \dots \dots (24)$$

If $r_j \in I_{25}$, then $\eta_{r_j} \in (w_j, \sigma(w_j))$, $[u_{r_j}, v_{r_j}] \subseteq [w_j, \sigma(w_j)]$ and $f^*(\eta_{r_j}) = f(w_j)$. Since $\{(w_j, [w_j, \sigma(w_j)])\}_{r_j \in I_{25}}$ is a δ -fine partial M_α -partition of $[a, b]_{\mathbb{T}}$, we have

$$\left| \sum_{r_j \in I_{25}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) - \sum_{r_j \in I_{25}} (M_\alpha^\Delta) \int_{w_j}^{\sigma(w_j)} f \right| \leq \epsilon \dots \dots (25)$$

By (11), (13), (21), (22), (24) and (25), we have

$$\begin{aligned} (II) &\leq \sum_{r_j \in I_{21}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) + \sum_{r_j \in I_{22}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) \\ &\quad + \sum_{r_j \in I_{23}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) + \sum_{r_j \in I_{241}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) \\ &\quad + \sum_{r_j \in I_{242}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) + \sum_{r_j \in I_{25}} f^*(\eta_{r_j})(\sigma(w_j) - w_j) \\ &\leq \sum_{r_j \in I_{21}} (M_\alpha^\Delta) \int_{w_j}^{\sigma(w_j)} f + \sum_{r_j \in I_{22}} (M_\alpha^\Delta) \int_{w_j}^{\sigma(w_j)} f + \sum_{r_j \in I_{23}} (M_\alpha^\Delta) \int_{w_j}^{\sigma(w_j)} f \\ &\quad + \sum_{r_j \in I_{241}} (M_\alpha^\Delta) \int_{w_j}^{\sigma(w_j)} f + \sum_{r_j \in I_{242}} (M_\alpha^\Delta) \int_{w_j}^{\sigma(w_j)} f \\ &\quad + \sum_{r_j \in I_{25}} (M_\alpha^\Delta) \int_{w_j}^{\sigma(w_j)} f + 20\epsilon \dots \dots (26) \end{aligned}$$

From (7), (23) and (26), we have

$$f^*(\mathcal{P}) < f^*(\mathcal{P}_1) + \epsilon \leq (M_\alpha^\Delta) \int_a^b f + 40\epsilon \dots \dots (27)$$

Similarly, we can show that

$$f^*(\mathcal{P}) > (M_\alpha^\Delta) \int_a^b f - 40\epsilon \dots \dots (28)$$

Hence, we have $|f^*(\mathcal{P}) - (M_\alpha^\Delta) \int_a^b f| < 40\epsilon$. Therefore f^* is M_α -integrable on $[a, b]$ and

$$(M_\alpha) \int_a^b f^* = (M_\alpha^\Delta) \int_a^b f.$$

□

THEOREM 2.5. *If $f^* : [a, b] \rightarrow \mathbb{R}$ is M_α -integrable on $[a, b]$, then $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is M_α^Δ -integrable on $[a, b]_{\mathbb{T}}$.*

Proof. Suppose that f^* is M_α -integrable on $[a, b]$. Define a function $g : [a, b] \rightarrow \mathbb{R}$ by

$$g(t) = \begin{cases} |f(\rho(t))| + |f(t)| & \text{if } t \in [a, b]_{\mathbb{T}}^{ls} \\ 0 & \text{if } t \in [a, b] - [a, b]_{\mathbb{T}}^{ls}. \end{cases}$$

Then since $[a, b]_{\mathbb{T}}^{ls}$ is a countable set, $g = 0$ a.e. on $[a, b]$. Hence, g is M_α -integrable on $[a, b]$ and $(M_\alpha) \int_a^b g = 0$. Let $\epsilon > 0$. Then there exists a gauge $\delta = (\delta_L, \delta_R)$ on $[a, b]$ such that

$$\left| f^*(D) - (M_\alpha) \int_a^b f^* \right| < \epsilon \quad \text{and} \quad |g(D)| < \epsilon$$

for each δ -fine M_α -partition D of $[a, b]$.

Define a Δ -gauge $\delta^1 = (\delta_L^1, \delta_R^1)$ on $[a, b]_{\mathbb{T}}$ by

$$\delta_L^1(t) = \delta_L(t) \quad \text{if } t \in [a, b]_{\mathbb{T}}$$

$$\delta_R(t) = \begin{cases} \delta_R(t) & \text{if } t \in [a, b]_{\mathbb{T}}^{rd} \\ \sigma(t) - t & \text{if } t \in [a, b]_{\mathbb{T}}^{rs}. \end{cases}$$

Assume that $D = \{(\xi_i, [u_i, v_i])\}_{i=1}^n$ is a δ^1 -fine M_α -partition of $[a, b]_{\mathbb{T}}$.

Let $A = \{i \leq n \mid \xi_i \in [a, b]_{\mathbb{T}}^{rs} \text{ and } [\xi_i, \sigma(\xi_i)] \subset [u_i, v_i]\}$,

$$B = \{1, 2, \dots, n\} - A.$$

For each $i \in A$, choose a δ -fine Henstock partition $D_i = \{(\xi_{ij}, [u_{ij}, v_{ij}])\}_{j=1}^{p_i}$ of $[\xi_i, \sigma(\xi_i)]$.

Let $D^* = \{(\xi_i, [u_i, v_i])\}_{i \in B} \cup \{(\xi_i, [u_i, \xi_i]) \mid i \in A, u_i < \xi_i\} \cup \left[\bigcup_{i \in A} D_i \right]$.

Then D^* is a δ -fine M_α -partition of $[a, b]$, and

$$\begin{aligned} & |f(D) - f^*(D^*)| \\ &= \left| \sum_{i \in A} f(\xi_i)(\sigma(\xi_i) - \xi_i) - \sum_{i \in A} \sum_{j \leq p_i} f^*(\xi_{ij})(v_{ij} - u_{ij}) \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \sum_{i \in A} \left[f(\xi_i)(\sigma(\xi_i) - \xi_i) - \sum_{\substack{j \leq p_i \\ \xi_{ij} < \sigma(\xi_i)}} f(\xi_i)(v_{ij} - u_{ij}) \right. \right. \\
&\quad \left. \left. - \sum_{\substack{i \leq p_i \\ \xi_{ij} = \sigma(\xi_i)}} f(\sigma(\xi_i))(v_{ij} - u_{ij}) \right] \right| \\
&= \left| \sum_{i \in A} \sum_{\substack{j \leq p_i \\ \xi_{ij} = \sigma(\xi_i)}} \left[f(\xi_i) - f(\sigma(\xi_i)) \right] (v_{ij} - u_{ij}) \right| \\
&\leq \sum_{i \in A} \sum_{\substack{j \leq p_i \\ \xi_{ij} = \sigma(\xi_i)}} \left[|f(\xi_i)| + |f(\sigma(\xi_i))| \right] (v_{ij} - u_{ij}) = g(D^*) < \epsilon.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left| f(D) - (M_\alpha) \int_a^b f^* \right| \\
&\leq |f(D) - f^*(D^*)| + \left| f^*(D^*) - (M_\alpha) \int_a^b f^* \right| < 2\epsilon.
\end{aligned}$$

Thus, f is M_α^Δ -integrable on $[a, b]_{\mathbb{T}}$ and

$$(M_\alpha^\Delta) \int_a^b f = (M_\alpha) \int_a^b f^*.$$

□

From Theorem 2.4 and 2.5, we get the following theorem.

THEOREM 2.6. *A function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is M_α^Δ -integrable $[a, b]_{\mathbb{T}}$ if and only if $f^* : [a, b] \rightarrow \mathbb{R}$ is M_α -integrable on $[a, b]$. In this case,*

$$(M_\alpha^\Delta) \int_a^b f = (M_\alpha) \int_a^b f^*.$$

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